Introduction to Mathematics and Modeling

lecture 4

Laws of differentiation

UNIVERSITY OF TWENTE.

academic year : 18-19
lecture : 4
build : December 3, 2018
slides : 24

Part 2

1. Directives concerning the MLP test
2. Section 3.3: laws of differentiation
3. Section 3.3: derivatives of exponential functions
4. Section 3.5: derivatives of trigonometric functions

- The test will take place in Therm.
- The test will consist of two parts: a written exam, and a digital test, using Chromebooks.
- You are not allowed to leave Therm before 9:15 (even when you are ready early).
- Be there *well* in time. If you use public transportation, take one bus or train earlier than usual.
  Although you can start late (but not later than 9:15), this should be an exception. Be well aware that you will disturb your fellow students that already started their test.
- You have maximal 30 minutes to complete the digital test (40 minutes for dislectic students).
-

- You can start the digital test between 8:45 and 9:15, **but the test shuts down at 9:45** (9:55 for dislectic students). This means that if you start the digital test after 9:15, you will not have the full 30/40 minutes at your disposal!
- The written exam starts at 8:45, and stops at 9:45 (9:55 for dislectic students).
- The use of an electronic calculator (or any other device) is not allowed. A calculator will be available on the chromebook as a separate app.
- For the second midterm test, a trigonometry formula sheet will **not** be issued.
- A practice tests (both digital and hand-written) will be published on Canvas.
- You can review the digital test no sooner than 12:00.

A wrong answer means 0 points!

- Use exact answers, avoid decimals if possible: ~~0.25~~ $\longrightarrow \frac{1}{4}$
- If decimals are required, use a decimal point: ~~$3,14$~~ $\longrightarrow 3.14$
- Simplify your answers as much as possible:

  ~~$\frac{2}{5} + \frac{1}{5}$~~ $\longrightarrow \frac{3}{5}$ ~~$3\frac{1}{2}$~~ $\longrightarrow \frac{7}{2}$

  ~~$\frac{6x^3}{3x^2}$~~ $\longrightarrow 2x$

  ~~$\frac{1}{2}\sqrt{4}$~~ $\longrightarrow 1$ ~~$\frac{1}{\sqrt{3}}$~~ $\longrightarrow \frac{1}{3}\sqrt{3}$

- Use the correct variable. If f is a function of t, i.e. $f(t) = t^2$, then

  ~~$f'(t) = 2x$~~ $\longrightarrow f'(t) = 2t$

- **Practice working with MyLabs Plus**

Basic functions:

- Power functions: x^α
- Trigonometric functions: $\sin(x), \cos(x), \tan(x)$
- Exponential functions: a^x, e^x
- Logarithms: $\log_a x, \ln x$ (next lecture)

Rules:

- Constant multiples
- Sums
- Products
- Quotients
- Compositions ('Chain Rule': next lecture)

The derivative of $f(x) = x^\alpha$

Let α be a real number and let $f(x) = x^\alpha$ then

$$f'(x) = \alpha x^{\alpha-1}$$

Examples:

$$f(x) = 1 = x^0 \quad \Longrightarrow \quad f'(x) = 0.x^{-1} = 0$$

$$f(x) = x = x^1 \quad \Longrightarrow \quad f'(x) = 1x^0 = 1$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \Longrightarrow \quad f'(x) = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f(x) = \frac{1}{x} = x^{-1} \quad \Longrightarrow \quad f'(x) = (-1)x^{-2} = -\frac{1}{x^2}$$

Rule of constant mutiplication

Let c be a constant, then

$$\frac{d}{dx}(c\,f(x)) = c\,f'(x).$$

Examples

- $$\frac{d}{dx}(2x^4) =$$
- $$\frac{d}{dx}(2x)^4 =$$
- $$\frac{d}{dx}\sqrt{3x} =$$

Sum rule

For all functions f and g we have

$$\frac{d}{dx}\Big(f(x)+g(x)\Big)=f'(x)+g'(x).$$

Difference rule

For all functions f and g we have

$$\frac{d}{dx}\Big(f(x)-g(x)\Big)=f'(x)-g'(x).$$

- $$\frac{d}{d\,x}\left(x^3 + \tfrac{4}{3}x^2 - 5x + 1\right) =$$

Theorem

If f and g are differentiable at x then

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x) + f(x)g'(x).$$

Example: differentiate $(x+1)(x-1)$.

- Apply the product rule:

  $$\frac{d}{dx}(x+1)(x-1) =$$

- Alternatively, expand $(x+1)(x-1) =$

  then differentiate:

Theorem

If g is differentiable at x, and $g(x) \neq 0$ then

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{g(x)^2}.$$

- Define $f(x) = 1/g(x)$, then

$$\begin{aligned} f(x)g(x) &= 1 \\ f'(x)g(x) + f(x)g'(x) &= 0 \\ f'(x)g(x) &= -f(x)g'(x) = -\frac{g'(x)}{g(x)} \\ f'(x) &= -\frac{g'(x)}{g(x)^2}. \end{aligned}$$

product rule

- Example: $\displaystyle \frac{d}{dx}\left(\frac{1}{x^2+1}\right) = -\frac{\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = -\frac{2x}{(x^2+1)^2}.$

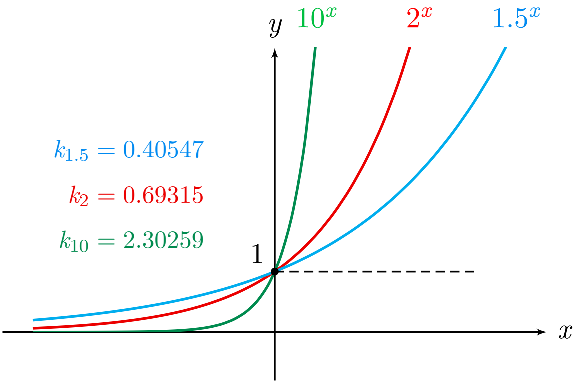
Theorem

If f and g are differentiable at x, and $g(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

- The quotient rule can be proven with the reciprocal rule and the product rule (see self-tuition exercises).
- Example:

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\right) =$$



- Define $k_a = f'(0)$ as the slope of the tangent line to the graph of $f(x) = a^x$ in the point $(0, 1)$.

Exponential Functions.nb

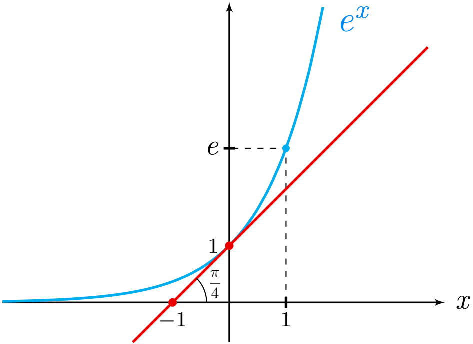
- Let $f(x) = a^x$, then

$$\begin{aligned} k_a &= f'(0) \\ &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{a^h - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h}. \end{aligned}$$

- For arbitrary x we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^h - 1}{h} a^x = k_a a^x = k_a f(x). \end{aligned}$$

- The derivative of an exponential function is *proportional* to the function.



- There is exactly one value of a for which $k_a = 1$.
- This value is denoted as e and it is approximately equal to 2.72.
- More precise

$$e \approx 2.7182818284590452353602874713527...$$

- The function e^x is called the **(natural) exponential function**.
- It has the elegant property

$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

The exponential function is it's own derivative!

- The number e is used as the base for the **natural logarithm**:

$$\ln(x) = \log_e(x).$$

- For the number k_a the following holds:

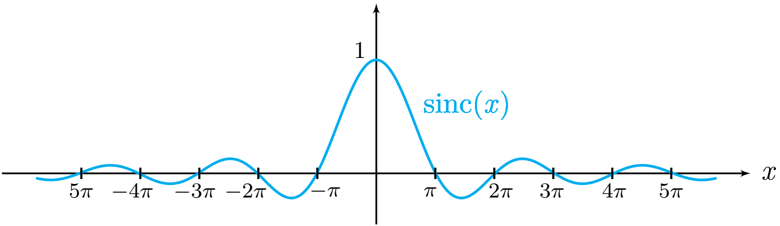
$$k_a = \ln a.$$

To prove this you need the *chain rule* (next lecture).

- $$\boxed{\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x}$$

Theorem – prelude to the Chain Rule

Prove that $\dfrac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$ *for all constants* a *and* b.



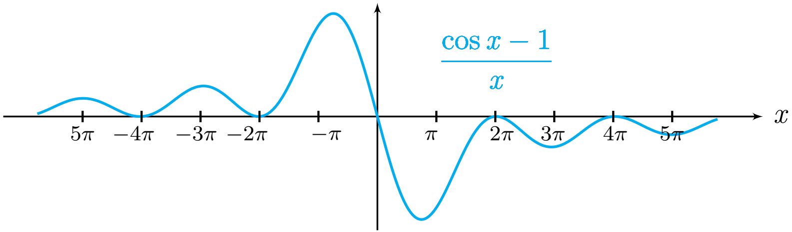
The **sample function** (denoted as sinc) is defined by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

- The sample function is continuous at 0:

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1.}$$

See self-tuition slides



Define g by

$$g(x) = \begin{cases} \dfrac{\cos x - 1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- The function g is continuous at 0:

$$\boxed{\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.}$$

See self-tuition slides

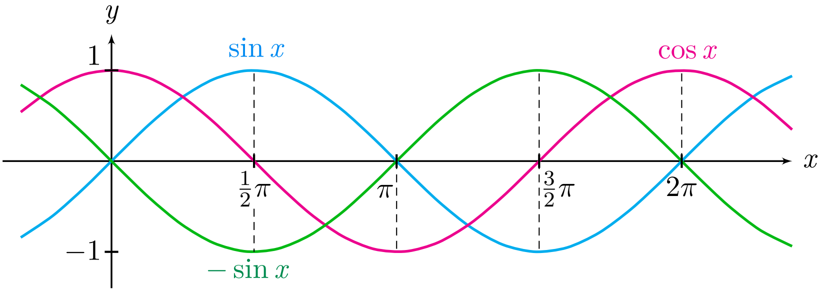
- $$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right)$$

$$= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\cdot 0+\cos(x)\cdot 1 = \boxed{\cos(x)}$$



$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$ and $$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

- For the derivative of $\cos x$, see the self-tuition exercises.

$$\frac{d}{d\,x}(\sin x) = \cos x$$

$$\frac{d}{d\,x}(\cos x) = -\sin x$$

$$\frac{d}{d\,x}(\tan x) = \frac{1}{\cos^2 x}$$

Learn them by heart!...

Example

Differentiate $e^x \sin(x)$.

Example

Differentiate $\sin(2x)$.